The Electric Potential:

To obtain the electric field we ~~solve~~ use Gauss's law which we have discussed in earlier notes. Since Gauss's law involves electric field which is a vector property and ~~to~~ to solve vector equations is more difficult than the scalar one. Therefore, ~~it~~ it will be useful to write Gauss's law in scalar form.

The electric field in terms of scalar potential is defined as

$$\vec{E} = -\nabla\phi \quad \text{——— (1)}$$

{ ∵ we have earlier showed $\nabla\times\vec{E} = 0$ for static electric field }

Now the work done in moving a charge from A to B in presence of electric field is given by

$$W = -\int_A^B \vec{F}\cdot d\vec{l}$$

$$= -\int_A^B (q_1\vec{E})\cdot d\vec{l} = q_1\int_A^B \nabla\phi\cdot d\vec{l}$$

or $W = q_1\left[\phi(\vec{x}_B) - \phi(\vec{x}_A)\right]$

Thus, the potential difference $\Delta\phi = \dfrac{W}{q_1}$ ——— (2)

~~Since~~

The Coulomb's law

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\,\rho(\vec{x}')\,d\vec{x}'$$

Since $\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\nabla \frac{1}{|\vec{x}-\vec{x}'|}$

$$\nabla \vec{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} \rho(\vec{x}')\, d\vec{x}'$$

$$-\nabla\phi = \frac{1}{4\pi\epsilon_0} \int \left[-\nabla \frac{1}{|\vec{x}-\vec{x}'|}\right] \rho(\vec{x}')\, d\vec{x}'$$

or $$\boxed{\phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d\vec{x}'} \quad \text{—— (3)}$$

Eq. (3) denotes the Coulomb's law in terms of scalar potential

~~Now~~ Next, $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

$$\nabla \cdot (-\nabla\phi) = \frac{\rho}{\epsilon_0}$$

or $$\boxed{\nabla^2\phi = -\frac{\rho}{\epsilon_0}} \quad \text{—— (4)}$$

Poisson equation

In eq. (4), if we consider ~~that~~ charge free case then,

$$\boxed{\nabla^2\phi = 0} \quad \text{—— (5)}$$

Eq. (5) is the Laplace equation.

Next, we can show that $\nabla \times \vec{E} = 0$ — (6) {using Eq. (1)}

we find the boundary condition using above expression

Integrating (6) over some arbitrary open surface

$$\int (\nabla \times \vec{E}) \cdot d\vec{a} = 0 \quad \text{or} \quad \int (\nabla \times \vec{E}) \cdot \hat{n}\, da = 0$$

using Stokes's theorem, we obtain

$$\oint \vec{E} \cdot d\vec{l} = 0 \longrightarrow$$ shows that the electrostatic fields are irrotational

Now if take line integral over a rectangular loop that straddles the material boundary surface and shrink it such that the contributions from the sides disappear. The contributions from the top and bottom becomes constant and we can take them out from the integral

$$\int_a^b \vec{E}_1 \cdot d\vec{l} + \int_b^a \vec{E}_2 \cdot d\vec{l} = 0$$

$$\vec{E}_1 \cdot \int_a^b d\vec{l} - \vec{E}_2 \int_a^b d\vec{l} = 0$$

or $\boxed{E_{1,T} = E_{2,T}}$ { $E_{1,T}$, $E_{2,T}$ tangential components of electric field to the surface

For the total electric field perpendicular to the surface normal vector, the above boundary condition can be expressed as

$$\boxed{\vec{E}_1 \times \hat{n} = \vec{E}_2 \times \hat{n}}$$